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Non-local symmetry and generating solutions for Harry–Dym-type equations

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Abstract. Non-local symmetries of the equations $u_0=f(u)u_{111}$, $w_0=g(w_{11})w_{111}$ are investigated. Equations which admit non-local linearization are described, and formulas for generating solutions derived. Non-Lie ansatz

$$u=h(x)\dot{\varphi}(\omega)+f(x)\varphi(\omega)+g(x)$$

is used for reduction of some nonlinear equations.

1. Introduction

Let us consider two classes of one-dimensional third-order nonlinear equations

$$u_0-f(u)u_{111}=0 \tag{1}$$

$$w_0-g(w_{11})w_{111}=0 \tag{2}$$

$$u_\mu=\partial_\mu u=\frac{\partial u}{\partial x_\mu} \qquad \underbrace{u_{1\ldots 1}}_{n}=\partial_1^n u=\frac{\partial^n u}{\partial x_1^n} \qquad w_\mu=\partial_\mu w=\frac{\partial w}{\partial x_\mu}$$

$$\underbrace{w_{1\ldots 1}}_{n}=\partial_1^n w=\frac{\partial^n w}{\partial x_1^n} \qquad (\mu=0,1,n\in N)$$

where $f(u)$, $g(w_{11})$ are arbitrary smooth functions.

In the present paper we pick out from the sets of equations (1) and (2) linearizable equations by means of non-local transformations. Also we investigate the non-Lie symmetry of (1) and (2) and obtain formulae of generating solutions for nonlinear equations belonging to the classes (1), (2). For reduction of (1) and (2) to ordinary differential equations (ODE) we use a non-Lie ansatz

$$u=h(x)\dot{\varphi}(\omega)+f(x)\varphi(\omega)+g(x) \qquad x=(x_0,x_1) \qquad \dot{\varphi}(\omega)=\frac{d\varphi}{d\omega} \tag{3}$$

which can be considered as a generalization of the ansatz [1, 2]

$$u=f(x)\varphi(\omega)+g(x).$$

Sets of partial exact solutions for nonlinear equations are constructed.

Note that equation (1) is equivalent to the equation

$$z_0 - \partial_1^3 c(z) = 0. \tag{4}$$

The connection between these equations is given by the transformation

$$c(z) = u. \tag{4a}$$

Thereby, the equality

$$f(u) = \dot{c}(c^{-1}[u])$$

holds. Here $c^{-1}[u]$ is the inverse function to $c(u)$. In the case $f(u) = u^3$, $c(z) = z^{-1/2}$, equation (4) coincides with the known Harry–Dym equation [3].

2. Non-local symmetry

Let us consider equation (4)

$$z_0 = \partial_1^3 c(z) = \partial_1^2(\dot{c}(z)z_1).$$

The substitution

$$z = w_{11} \tag{5a}$$

transforms (4) to the equation

$$w_0 = \dot{c}(w_{11})w_{111}. \tag{5}$$

Using the Euler–Ampere transformation

$$w = y_1 v_1 - v \qquad x_1 = v_1 \qquad x_0 = y_0 \qquad v = v(y_0, y_1) \qquad v_{11} \neq 0 \tag{6}$$

for (5), we obtain

$$v_0 = \dot{c}(v_{11}^{-1})v_{11}^{-3}v_{111}. \tag{7}$$

Using the substitution

$$v_{11} = z(y_0, y_1) \tag{7a}$$

in (7), twice differentiated with respect to y_1, we obtain

$$z_0 = \partial_1^2(\dot{c}(z^{-1})z^{-3}z_1). \tag{8}$$

It follows from (8), that transformations (5a), (6), (7a) do not take out any equation (4) beyond this class of equations; none the less the set of equations (4) is not invariant under these transformations. If function $\dot{c}(z^{-1})z^{-3}$ in (8) satisfies the condition

$$\dot{c}(z^{-1})z^{-3} = \lambda \qquad \lambda = \text{constant} \tag{9a}$$

then (4) is linearizable. When the condition

$$\dot{c}(z) = \dot{c}(z^{-1})z^{-3} \tag{9b}$$

holds, equation (8) coincides with the initial equation (4), i.e. these equations are invariant with respect to non-local transformations (5a), (6), (7a).

The condition (9b) allows one to describe all equations of class (4) which are invariant with respect to the transformations (5a), (6), (7a).

Theorem 1. Equation (4) is invariant with respect to the transformations (5*a*), (6), (7*a*), if it is of the form

$$z_0 = \partial_1^2[z^{-3/2}\varphi(\ln z)z_1]. \tag{10}$$

Here $\varphi(\alpha)$ is an arbitrary smooth even function.

Corollary 1. Equation (4) is invariant with respect to transformations (4*a*), (5*a*), (6), (7*a*), (4*a*) if it has the form

$$u_0 = (c^{-1}[u])^{-3/2}\varphi(\ln c^{-1}[u])u_{111} \tag{11}$$

where $c^{-1}[u]$ is the inverse function to $c(u)$, and it can be determined implicitly from the formula

$$u = \int z^{-3/2}\varphi(\ln z)\,\mathrm{d}z. \tag{12}$$

Example 1. From theorem 1 and corollary 1 under $\varphi(\alpha)=1$ we obtain the following invariant equations:

$$z_0 = \partial_1^2(-\tfrac{1}{2}z^{-3/2}z_1) = \partial_1^3(z^{-1/2}) \tag{13}$$

$$u_0 = u^3 u_{111}. \tag{14}$$

Equation (13) is known as the Harry–Dym equation. Putting $\varphi(\alpha)=\cos\alpha$ we obtain the equation

$$z_0 = \partial_1^2(z^{-3/2}\cos\{\ln z\}z_1) \tag{15}$$

and the corresponding equation of class (1)

$$u_0 = (c^{-1}[u])^{-3/2}\cos\ln(c^{-1}[u])u_{111}. \tag{16}$$

Here $c^{-1}[u]$ is determined implicitly by the formula

$$u = \tfrac{4}{5}[\sin\ln z - \tfrac{1}{2}\cos\ln z]z^{-1/2}. \tag{17}$$

So we find that the equations

$$u_0 = u^{3/2}u_{111} \tag{18a}$$

$$z_0 = \partial_1^3(z^{-2}) \tag{18b}$$

$$w_0 = w_{11}^{-3}w_{111} \tag{18c}$$

are reduced to the linear equation

$$v_0 = v_{111} \qquad (\lambda = 1) \tag{19}$$

and that, in particular, the Harry–Dym equation and the equations which are connected with it

$$u_0 = u^3 u_{111} \tag{20a}$$

$$z_0 = \partial_1^3(z^{-1/2}) \tag{20b}$$

$$w_0 = w_{11}^{-3/2}w_{111} \tag{20c}$$

are invariant with respect to the corresponding non-local transformations.

3. The non-local superposition and generating solutions

Theorem 2. The superposition formula for solutions of (18*a*)

$$u_0 = u^{3/2} u_{111} \tag{18a}$$

has the form

$$\begin{cases} \overset{(3)}{u}(x_0, x_1) = \overset{(1)}{u}(x_0, \overset{(1)}{\tau}) + \overset{(2)}{u}(x_0, \overset{(2)}{\tau}) + 2\sqrt{\overset{(1)}{u}(x_0, \overset{(1)}{\tau})\overset{(2)}{u}(x_0, \overset{(2)}{\tau})} & (21a) \\ \mathrm{d}\overset{(1)}{\tau} \Big/ \sqrt{\overset{(1)}{u}(x_0, \overset{(1)}{\tau})} = \mathrm{d}\overset{(2)}{\tau} \Big/ \sqrt{\overset{(2)}{u}(x_0, \overset{(2)}{\tau})} & (21b) \\ \overset{(1)}{\tau} + \overset{(2)}{\tau} = x_1 & (21c) \\ \overset{(1)}{\tau_0} = \dfrac{1}{2} \dfrac{\sqrt{\overset{(1)}{u}(x_0, \overset{(1)}{\tau})\overset{(2)}{u}(x_0, \overset{(2)}{\tau})}}{\sqrt{\overset{(1)}{u}(x_0, \overset{(1)}{\tau})} + \sqrt{\overset{(2)}{u}(x_0, \overset{(2)}{\tau})}} [\overset{(1)}{u}_{11}(x_0, \overset{(1)}{\tau}) + \overset{(2)}{u}_{11}(x_0, \overset{(2)}{\tau})]. & (21d) \end{cases}$$

Let us illustrate the efficiency of formula (21).

Example 2. Let us take the simplest stationary solutions of (18*a*):

$$\overset{(1)}{u}(\overset{1}{x}_1) = (\overset{1}{x}_1)^2 \qquad \overset{(2)}{u}(\overset{2}{x}_1) = (\overset{2}{x}_1)^2.$$

Let us replace $\overset{1}{x}_1$ and $\overset{2}{x}_1$ in these solutions by parameters $\overset{(1)}{\tau}$, $\overset{(2)}{\tau}$:

$$\overset{(1)}{u} = \overset{(1)}{\tau}{}^2 \qquad \overset{(2)}{u} = 4\overset{(2)}{\tau}{}^2.$$

The differential equation (21*b*) takes the form

$$\mathrm{d}\overset{(2)}{\tau} / \mathrm{d}\overset{(1)}{\tau} = 2(\overset{(2)}{\tau} / \overset{(1)}{\tau}) \tag{22}$$

and has the general solution

$$\overset{(2)}{\tau} = -\frac{\overset{(1)}{\tau}{}^2}{2\lambda(x_0)}. \tag{23}$$

Here $\lambda(x_0)$ is an arbitrary smooth function. The equation for $\overset{(1)}{\tau}$

$$\overset{(1)}{\tau}{}^2 - 2\lambda\overset{(1)}{\tau} + 2\lambda x_1 = 0 \tag{24}$$

we obtain by means of (21*c*), replacing $\overset{(2)}{\tau}$ in (23) by the expression $x - \overset{(1)}{\tau}$. From (24) we find

$$\begin{aligned} \overset{(3)}{u}(x_0, x_1) &= (\overset{(1)}{\tau} + 2\overset{(2)}{\tau})^2 = (2x_1 - \overset{(1)}{\tau})^2 \\ &= [2x_1 - \lambda \pm \sqrt{\lambda^2 - 2x_1\lambda}]^2. \end{aligned} \tag{25}$$

The function $\lambda(x_0)$ can be defined more precisely from the condition that $\overset{(1)}{\tau}$ is a solution of (21d). As a result we obtain an equation for $\lambda(x_0)$

$$\dot{\lambda} = -6\lambda.$$

Therefore

$$\lambda = c\exp(-6x_0)$$

where c is an arbitrary constant. So a new solution $\overset{(3)}{u}$, which is constructed from $\overset{(1)}{u}$ and $\overset{(2)}{u}$ is of the form

$$\overset{(3)}{u}(x_0, x_1) = [2x_1 - c\exp(-6x_0) \pm \sqrt{c^2\exp(-12x_0) - 2cx_1\exp(-6x_0)}]^2. \tag{26}$$

Example 3. Let us choose the following two solutions of (18a):

$$\overset{(1)}{u} = x_1^2 \qquad \overset{(2)}{u} = 9x_1^2$$

and rewrite them in variables $\overset{(1)}{\tau}$ and $\overset{(2)}{\tau}$

$$\overset{(1)}{u} = \overset{(1)}{\tau}{}^2 \qquad \overset{(2)}{u} = 9\overset{(2)}{\tau}{}^2.$$

Unlike the previous example from the ODE (21b), a cubic equation for $\overset{(1)}{\tau}$ is obtained:

$$\overset{(1)}{\tau}{}^3 - \lambda\overset{(1)}{\tau} + \lambda x_1 = 0 \qquad \lambda = \lambda(x_0). \tag{27}$$

The real solution of (27) can be written in the form

$$\overset{(1)}{\tau} = -3\lambda^{-1}\cos\tfrac{1}{3}\cos^{-1}\lambda x_1 \tag{27a}$$
$$\lambda = \tfrac{3}{2}\sqrt{3}\lambda^{-1/2}(x_0).$$

The solution $\overset{(3)}{u}$

$$\overset{(3)}{u}(x_0, x_1) = (3x_1 - 2\overset{(1)}{\tau})^2 = 9[x_1 - \tfrac{2}{3}\overset{(1)}{\tau}]^2$$
$$= 9[x_1 + 2\lambda^{-1}\cos\tfrac{1}{3}\cos^{-1}\lambda x_1]^2 \tag{28}$$

we find from formula (21a). The condition on $\lambda(x_0)$ is of the form

$$\dot{\lambda} = 12\lambda.$$

Hence

$$\lambda = c\exp(12x_0)$$

where c is an arbitrary constant. Finally, one can write solution $\overset{(3)}{u}$ in the form

$$\overset{(3)}{u}(x_0, x_1) = 9[x_1 + 2c\exp(-12x_0)\cos\tfrac{1}{3}\cos^{-1}\{cx_1\exp(12x_0)\}]^2. \tag{29}$$

4. The non-group generating of solutions

For equations of class (11) we can write a formula for generating solutions. Let $\overset{(1)}{u}(x_0, x_1)$ be a known partial solution of nonlinear equation (11), and $\overset{(2)}{u}(x_0, x_1)$ its new solution; then the following assertion holds true:

Theorem 3. The formula for generating solutions of equation (11), connected with the non-local symmetry (4*a*), (5*a*), (6), (7*a*), (4*a*), has the form

$$\begin{cases} \overset{(2)}{u}(x_0, x_1) = \left[x_1\tau - \int \left(\int \overset{(1)}{u}{}^{-2}(x_0, \tau)\, d\tau \right) d\tau \right]^{1/2} & (30a) \\ \qquad = \overset{(1)}{u}{}^{-1}(x_0, \tau) & (30b) \\ x_1 = \int \overset{(1)}{u}{}^{-2}(x_0, \tau)\, d\tau & (30c) \\ \tau_0 - \partial_1(\tau_1^{-3/2}\tau_{11}) = 0. & (30d) \end{cases}$$

Let us demonstrate the efficiency of formula (30) for equation (20*a*) on several simple examples.

Example 4. Let $\overset{(1)}{u} = 1$. Then

$$\overset{(2)}{u}(x_0, \tau) = \left[x_1\tau - \int \left(\int d\tau \right) d\tau \right]^{1/2} \qquad x_1 = \int d\tau = \tau + \lambda_1(x_0).$$

where $\lambda_1(x_0)$ is an arbitrary function. Calculating the integral in the first equality and resolving the second one with respect to τ, we obtain

$$\overset{(2)}{u}(x_0, \tau) = [x_1\tau - \tfrac{1}{2}\tau^2 - \lambda_1\tau + \lambda_2(x_0)]^{1/2} \tag{31}$$

$$\tau = x_1 - \lambda_1(x_0). \tag{32}$$

Having excluded parameter τ from equalities of the system (31), (32), we obtain the solution $\overset{(2)}{u}(x_0, x_1)$ in explicit form

$$\overset{(2)}{u}(x_0, x_1) = [\lambda_2 - \tfrac{1}{2}\lambda_1^2]^{1/2} \equiv \lambda_3 = \text{constant}. \tag{33}$$

Example 5. The function

$$\overset{(1)}{u}(x_0, x_1) = \tfrac{1}{4}(\lambda_1 - x_1)^2 \tag{34}$$

is the solution of (20*a*), where λ_1 is an arbitrary constant. It follows from relations (30*b*, *c*), that

$$\overset{(2)}{u}(x_0, \tau) = 4(\lambda_1 - \tau)^{-2} \tag{35}$$

$$x_1 = \tfrac{16}{3}(\lambda_1 - \tau)^{-3} + \lambda_2(x_0). \tag{36}$$

Resolving (36) with respect to τ, one obtains

$$\begin{aligned} &\tau = h[x_1 - \lambda_2(x_0)]^{-1/3} + \lambda_1 \\ &h = -(\tfrac{3}{16})^{-1/3}. \end{aligned} \tag{37}$$

Substituting τ from formula (37) into condition (30d), one obtains

$$\dot{\lambda}_2 = -1.$$

Let us substitute specified value of τ into formula (35) and find the solution $\overset{(2)}{u}$

$$\overset{(2)}{u}(x_0, x_1) = k(x_0 + x_1)^{2/3} \qquad k = (\tfrac{3}{2})^{2/3}. \tag{38}$$

5. Non-Lie ansätze

Let us consider the ansatz of the form

$$w = h(x)\dot{\varphi}(\omega) + f(x)\varphi(\omega) + g(x)$$
$$x = (x_0, x_1) \qquad \dot{\varphi}(\omega) = \frac{\mathrm{d}\varphi}{\mathrm{d}\omega} \tag{39}$$

for constructing solutions of (20c):

$$w_0 - w_{11}^{-3/2} w_{111} = 0. \tag{20c}$$

Let us summarize the results obtained for equation (20c) in table 1.
The ansätze 1–3 reduce the PDE (20c) to the following ODEs:

$$\text{(i)}\ \dot{\varphi} = 0 \qquad \dot{\lambda}_1 = -4\varphi \qquad \dot{\lambda}_2 = 0$$
$$\text{(ii)}\ [2x_1\partial_\omega - 1][2\ddot{\varphi}^{-1/2} - \dot{\varphi}] = 0 \qquad \dot{\lambda}_1 = \dot{\lambda}_2 = 0$$
$$\text{(iii)}\ [\partial_\omega - 2x_1][\tfrac{2}{3}\varphi + \dot{\varphi} - 2\ddot{\varphi}^{-1/2}] = 0 \qquad \dot{\lambda}_1 = \dot{\lambda}_2 = 0.$$

Before reducing (20c) by means of ansatzes 4–7 let us make the substitution

$$(\ddot{\varphi}(\omega))^{-1/2} = \psi(\omega).$$

As a result we obtain other reduced ODEs:

$$\text{(iv)}\ \tfrac{1}{3}\psi - \dot{\psi} + \psi^3(4\dot{\psi} - \ddot{\psi}) = 0$$
$$\text{(v)}\ \tfrac{1}{3}\psi - \dot{\psi} + \psi^3(4\dot{\psi} + \ddot{\psi}) = 0$$
$$\text{(vi)}\ \dot{\psi} - \psi^3(4\dot{\psi} - \ddot{\psi}) = 0$$
$$\text{(vii)}\ \dot{\psi} + \psi^3(4\dot{\psi} + \ddot{\psi}) = 0.$$

It is known [1] that an infinitesimal operator

$$X = \xi^i(x, w)\partial_i + \eta(x, w)\partial_w \qquad (i = \overline{1, n})$$

which generates a Lie ansatz, corresponds to the equation

$$Q[w] = \xi^i(x, w)w_i - \eta(x, w) = 0. \tag{40}$$

Equations of the form (40) correspond to non-Lie ansätze 1–7:

$$\text{(i)}\ x_1 w_{111} + 4w_{11} = 0$$
$$\text{(ii)}\ w_{011} + x_1^2 w_{111} + 4x_1 w_{11} = 0$$
$$\text{(iii)}\ x_0 w_{011} + x_1^2 w_{111} + 4(x_1 - \tfrac{1}{6})w_{11} = 0$$
$$\text{(iv)}\ x_0 w_{011} + (x_1^2 - 1)w_{111} + 4(x_1 - \tfrac{1}{6})w_{11} = 0$$

Table 1. Non-Lie ansätze of the form (39) $w=h(x)\dot{\varphi}(\omega)+f(x)\varphi(\omega)+g(x)$ for equation $w_0=w_{11}^{-3/2}w_{111}$.

N	ω	$h(x)$	$f(x)$	$g(x)$
1	x_0	0	$\frac{x_1^{-2}}{6}\varphi^{-3}(\omega)$	$\lambda_1 x_1+\lambda_2$
2	$x_0+x_1^{-1}$	1	$-2x_1$	$\lambda_1 x_1+\lambda_2$
3	$\ln x_0+x_1^{-1}$	$x_0^{-1/3}$	$-2x_1x_0^{-1/3}$	$\lambda_1 x_1+\lambda_2$
4	$\ln x_0+\tanh^{-1}x_1$	$-x_0^{-1/3}$	$2x_1x_0^{-1/3}$	$x_0^{-1/3}\Big\{-2\int\varphi(\omega)\,\mathrm{d}x_1+2\int\varphi(\omega)\times\frac{(x_1^2+1)}{(x_1^2-1)}\mathrm{d}x_1+8\int\Big(\int\varphi(\omega)\times\frac{x_1}{(x_1^2-1)^2}\mathrm{d}x_1\Big)\mathrm{d}x_1+\lambda_1x_1+\lambda_2$
5	$\ln x_0-\tan^{-1}x_1$	$x_0^{-1/3}$	$-2x_1x_0^{-1/3}$	$x_0^{-1/3}\Big\{2\int\varphi(\omega)\,\mathrm{d}x_1-2\int\varphi(\omega)\times\frac{(1-x_1^2)}{(1+x_1^2)}\mathrm{d}x_1-8\int\Big(\int\varphi(\omega)\times\frac{x_1}{(x_1^2+1)^2}\mathrm{d}x_1\Big)\mathrm{d}x_1+\lambda_1x_1+\lambda_2$
6	$x_0+\tanh^{-1}x_1$	1	$-2x_1$	$-2\int\varphi(\omega)\,\mathrm{d}x_1+2\int\varphi(\omega)\,\frac{1+x_1^2}{1-x_1^2}\mathrm{d}x_1-8\int\Big(\int\varphi(\omega)\,\frac{x_1}{(1-x^2)^2}\mathrm{d}x_1\Big)\mathrm{d}x_1+\lambda_1x_1+\lambda_2$
7	$x_0-\tan^{-1}x_1$	1	$-2x_1$	$-2\int\varphi(\omega)\,\mathrm{d}x_1-2\int\varphi(\omega)\,\frac{1-x_1^2}{1+x_1^2}\mathrm{d}x_1+8\int\Big(\int\varphi(\omega)\,\frac{x_1}{(1+x_1^2)^2}\mathrm{d}x_1\Big)\mathrm{d}x_1+\lambda_1x_1+\lambda_2$

(v) $x_0w_{011}+(x_1^2+1)w_{111}+4(x_1-\frac{1}{6})w_{11}=0$

(vi) $w_{011}+(x_1^2-1)w_{111}+4x_1w_{11}=0$

(vii) $w_{011}+(x_1^2+1)w_{111}-4x_1w_{11}=0.$

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